Tiling a rectangle with squares

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1 Introduction

This paper contains a short, informal solution to the following problem.

Main problem

If the sides of a rectangle are rational multiples of one another, it's not hard to see that the rectangle can be tiled with squares. (After all, scaling the width and the height of the rectangle by some appropriate constant yields a rectangle with integer sides, which can be tiled with unit squares.) Are these the only rectangles that can be tiled with squares?

As it turns out, the answer to this question is yes: if a rectangle can be tiled with squares, then the sides must be rational multiples of one another. We will prove this using undergraduate linear algebra (at the level of the course Linear Algebra II taught at Leiden University).

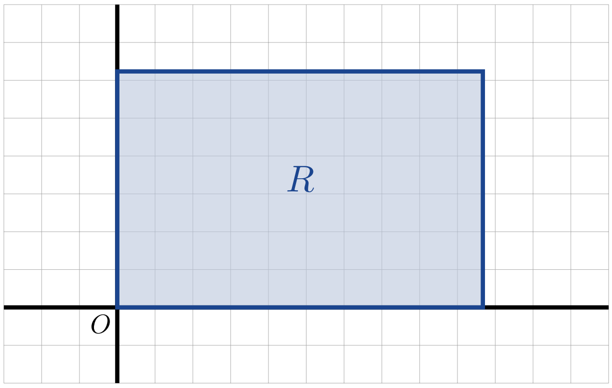
The present exposition is mainly intended as material for a talk in the second year course SPC ("Seminarium Presenteren en Communiceren") taught at Leiden University. In particular, we will only use techniques from undergraduate linear algebra. Other solutions to the problem exist; see Further reading.

There may well be more material in this paper than can reasonably be covered in a single SPC presentation. It is up to the speaker to make a selection.

As the material connects to some of the topics treated in the course Linear Algebra II, this topic should not be chosen for a presentation earlier than mid November.

2 Notation

Let us encode a rectangle as the subset $R \subseteq \mathbb{R}^2$ consisting of all points in or on the rectangle (border + interior). Without loss of generality, we will assume the sides of the main rectangle to be parallel to the coordinate axes. Furthermore, we assume the lower left corner of the rectangle to coincide with the origin.



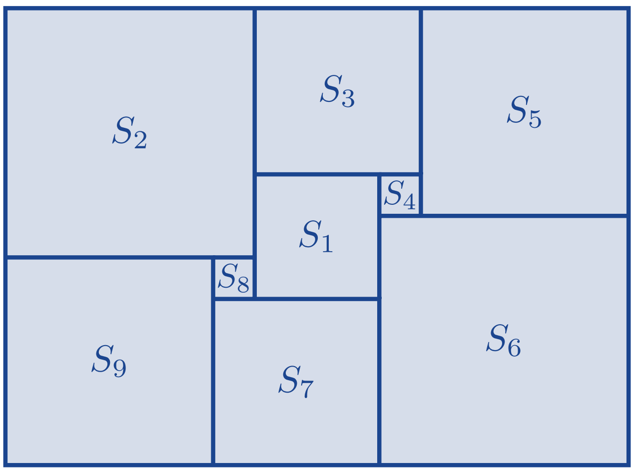
In this paper, we say that a *tiling* of R is a finite collection of non-degenerate (i.e. positive area) rectangles $S_1, \ldots, S_n \subseteq R$ satisfying the following two conditions:

- The union $S_1 \cup \cdots \cup S_n$ is equal to R;
- For any $i, j \in \{1, \ldots, n\}$ with $i \neq j$, the intersection $S_i \cap S_j$ has zero area (in other words, it is either empty or merely a single line segment or point).

Note that the sides of the rectangles in a tiling must all be parallel to the coordinate axes: we only have angles of 90° at our disposal. If there exists a tiling of the rectangle R such that every S_i is a square, we say that *R can be tiled with squares.*

A rectangle with integer sides can always be tiled with squares: we can simply lay out a grid of 1×1 squares. This solution can then be scaled to tile any rectangle with rational sides, or indeed any rectangle where the sides are rational multiples of one another.

A non-trivial example of a tiling is given in the figure below.



Let us denote the width and height of a given rectangle R as $w(R)$ and $h(R)$, respectively. If R is a square, then these two lengths coincide. In this case we write $\ell(R)$ for this common length. In the above example we have

$$\begin{aligned} \ell(S_1) &= 3, \\ \ell(S_2) = \ell(S_6) &= 6, \\ \ell(S_3) = \ell(S_7) &= 4, \\ \ell(S_4) = \ell(S_8) &= 1, \\ \ell(S_5) = \ell(S_9) &= 5. \end{aligned}$$

3 Area functions

From now on, let us assume that R is a fixed rectangle and that $\mathcal{T} = \{S_1, \ldots, S_n\}$ is a fixed tiling of R. Now we are going to define a vector space corresponding to $\mathcal{T}$. First, note that $\mathbb{R}$ is not only a vector space over the ground field $\mathbb{R}$, but can also be considered as a vector space over $\mathbb{Q}$. The addition is still the same as before, but now we only have rational scalars. (This idea is known as *restriction of scalars*.) Something strange happens to the dimension: whereas $\mathbb{R}$ is one-dimensional as a real vector space, it is in fact infinite-dimensional as a vector space over $\mathbb{Q}$. One way to prove this: a finite-dimensional rational vector space is isomorphic to $\mathbb{Q}^d$ for some positive integer d, but this is a countable set (whereas $\mathbb{R}$ is uncountable).

Since we don't have the tools yet to deal with infinite-dimensional vector spaces, we have to somehow throw away a lot of information. To that end, let us define $\mathscr{L}_\mathcal{T} \subseteq \mathbb{R}$ as the $\mathbb{Q}$-subspace of $\mathbb{R}$ spanned by $w(S_1), h(S_1), \ldots, w(S_n), h(S_n)$. In other words, we have

$$\mathscr{L}_\mathcal{T} = \big\{\lambda_1 w(S_1) + \mu_1 h(S_1) + \cdots + \lambda_n w(S_n) + \mu_n h(S_n) \quad : \quad \lambda_1, \mu_1, \ldots, \lambda_n, \mu_n \in \mathbb{Q}\big\}.$$

Observe that the dimension of $\mathscr{L}_\mathcal{T}$ is at most $2n$, since the space is spanned by $2n$ vectors. (Note: we are talking here about the dimension of $\mathscr{L}_\mathcal{T}$ as a vector space over $\mathbb{Q}$. This should be clear from the context, since $\mathscr{L}_\mathcal{T}$ is not a vector space over $\mathbb{R}$.) In particular, $\mathscr{L}_\mathcal{T}$ is finite-dimensional. An immediate consequence is that $\mathscr{L}_\mathcal{T}$ is countable, but we do not need that. It turns out that this finite-dimensional rational vector space $\mathscr{L}_\mathcal{T}$ contains all the information we need to study the tiling $\mathcal{T}$.

Definition 3.1. A $\mathcal{T}$*-area function*[1] is a $\mathbb{Q}$-bilinear map from $\mathscr{L}_\mathcal{T} \times \mathscr{L}_\mathcal{T}$ to some rational vector space V. (In particular, any bilinear form $\mathscr{L}_\mathcal{T} \times \mathscr{L}_\mathcal{T} \to \mathbb{Q}$ is a $\mathcal{T}$-area function.)

The name of these functions is explained by the following lemma.

Lemma 3.2. *Let $f : \mathscr{L}_\mathcal{T} \times \mathscr{L}_\mathcal{T} \to V$ be an area function. Then we have*

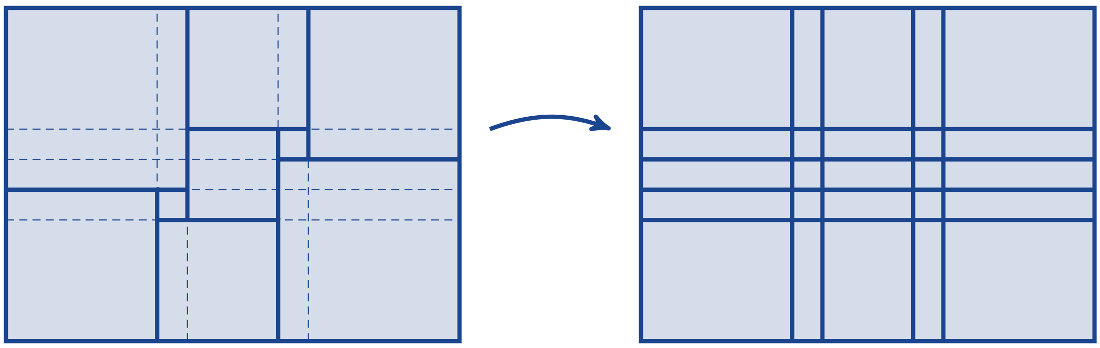
$$f\big(w(R), h(R)\big) = \sum_{i=1}^{n} f\big(w(S_i), h(S_i)\big).$$

Before going into the proof, let us pause for a moment in order to try to understand what this lemma says. Intuitively, we will think of $f(w(R), h(R))$ as the area of R, and of $f(w(S_i), h(S_i))$ as the area of S_i. However, this is not your standard notion of area, but a slight generalisation that turns out to work well in the context of rectangular tilings. We recover the standard notion of area by taking the bilinear form $f : \mathscr{L}_\mathcal{T} \times \mathscr{L}_\mathcal{T} \to \mathbb{R}$ given by $f(x, y) = x \cdot y$.

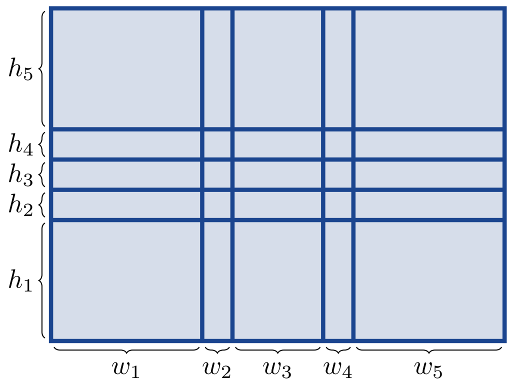
[1] Terminology invented by the author.

The lemma states that the (generalised) area of R is equal to the sum of the (generalised) areas of $S_1, \dots, S_n$. Note that the equation in Lemma 3.2 is well defined: we have $w(S_i), h(S_i) \in \mathscr{L}_\mathcal{T}$ by definition, and we have $w(R), h(R) \in \mathscr{L}_\mathcal{T}$ since both can be written as the sum of some of the $w(S_1), h(S_1), \dots, w(S_n), h(S_n)$. (For instance, in the example from before we may write $w(R) = w(S_2) + w(S_3) + w(S_5)$ and $h(R) = h(S_3) + h(S_1) + h(S_7)$.) As such, $\mathscr{L}_\mathcal{T}$ is the smallest rational vector subspace of $\mathbb{R}$ so that the equation in lemma Lemma 3.2 is well defined.

Proof of Lemma 3.2. Let us extend the edges of the squares $S_1, \dots, S_n$ to form line segments spanning the entire width or height of the rectangle R. This way we obtain a tiling of R as an irregular grid of rectangles, as illustrated in the figure below.



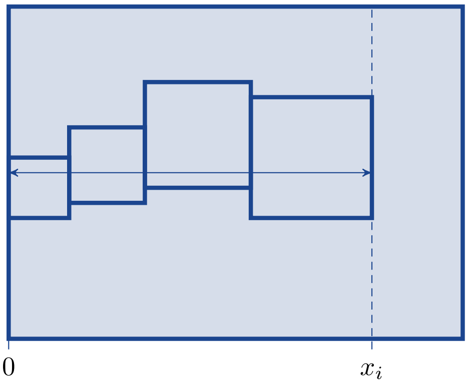
We call the rectangles in this irregular grid *elementary rectangles*. Furthermore, let us say that the grid has r rows and c columns, and let $w_1, \dots, w_c$ and $h_1, \dots, h_r$ denote the widths and heights of these columns and rows, respectively.



We shall prove that $w_1, \dots, w_c \in \mathscr{L}_\mathcal{T}$ holds. Let $x_0, \dots, x_c$ denote the x-coordinates of the vertical lines (in increasing order), so that we have $w_i = x_i - x_{i-1}$. Recall that we have $x_0 = 0$ by assumption (the lower right corner of R is the origin). Now we prove that every x_i can be written as the sum of some $w(S_1), \dots, w(S_n)$:

- Choose some y_i such that the point (x_i, y_i) lies on the left or right side of some rectangle S_j from the tiling $\mathcal{T}$, but not on one of the $r+1$ vertical lines.
- Draw the horizontal line segment connecting (x_i, y_i) with the point $(0, y_i)$. This line passes through an integer number of rectangles from the tiling $\mathcal{T}$. This leaves us with an expression of x_i as a sum of some of the $w(S_1), \dots, w(S_n)$.

This technique is illustrated in the figure below (on the next page).



Therefore we have $x_0, \ldots, x_c \in \mathscr{L}_\mathcal{T}$. Every w_i is the difference of two of these, so we also have $w_1, \ldots, w_c \in \mathscr{L}_\mathcal{T}$. An analogous proof shows that $h_1, \ldots, h_r \in \mathscr{L}_\mathcal{T}$ holds as well.

Now we can do algebra. By the bilinearity of f, we have

$$\begin{aligned} f\big(w(R)\,,\, h(R)\big) &= f\big(w_1 + \cdots + w_c\,,\, h_1 + \cdots + h_r\big) \\ &= f\big(w_1 + \cdots + w_c\,,\, h_1\big) \;+\; \cdots \;+\; f\big(w_1 + \cdots + w_c\,,\, h_r\big) \qquad (3.3) \\ &= \sum_{i=1}^{c} \sum_{j=1}^{r} f\big(w_i\,,\, h_j\big). \end{aligned}$$

Geometrically, this means that the (generalised) area of the rectangle R is equal to the sum of the (generalised) areas of all elementary rectangles. By the same line of reasoning, the (generalised) area of a rectangle S_i in the tiling $\mathcal{T}$ can also be written as the sum of the (generalised) areas of the elementary rectangles it contains. Every elementary rectangle is contained in precisely one rectangle from the tiling $\mathcal{T}$, so we find

$$\sum_{i=1}^{n} f\big(w(S_i), h(S_i)\big) = \sum_{i=1}^{c} \sum_{j=1}^{r} f\big(w_i, h_j\big).$$

Combining this with (3.3) gives the desired result. ■

4 Solving the main problem

As mentioned before, any bilinear form $\mathscr{L}_\mathcal{T} \times \mathscr{L}_\mathcal{T} \to \mathbb{Q}$ is a $\mathcal{T}$-area function. In particular, if we have a linear form $\varphi : \mathscr{L}_\mathcal{T} \to \mathbb{Q}$, then it is easy to see that the map $f : \mathscr{L}_\mathcal{T} \times \mathscr{L}_\mathcal{T} \to \mathbb{Q}$ given by $f(x, y) = \varphi(x) \cdot \varphi(y)$ is a bilinear form. It is this construction that leads to a solution of the main problem. Most of the magic happens in the following lemma.

Lemma 4.1. *Suppose that our tiling $\mathcal{T}$ consists of squares, and that $\varphi : \mathscr{L}_\mathcal{T} \to \mathbb{Q}$ is a linear map satisfying $\varphi(w(R)) = 0$. Then we have $\varphi = 0$.*

Proof. Consider the $\mathcal{T}$-area function $f : \mathscr{L}_\mathcal{T} \times \mathscr{L}_\mathcal{T} \to \mathbb{Q}$ given by $f(x, y) = \varphi(x) \cdot \varphi(y)$. Then, by Lemma 3.2 we have

$$\sum_{i=1}^{n} \varphi\big(\ell(S_i)\big)^2 = \sum_{i=1}^{n} f\big(w(S_i), h(S_i)\big) = f\big(w(R), h(R)\big) = \varphi(w(R)) \cdot \varphi(h(R)) = 0.$$

Since every term $\varphi(\ell(S_i))^2$ is non-negative, it follows that $\varphi(\ell(S_i)) = 0$ must hold for all i. Now, as $\mathscr{L}_\mathcal{T}$ is spanned by $\ell(S_1), \ldots, \ell(S_n)$, we must have $\varphi = 0$. ■

Observe what happens here: the φ-area of a square (geometric figure) is a square (non-negative real number), so it must be zero if the total area of the rectangle R is zero! This small piece of non-linear algebra has great consequences for the vector space $\mathscr{L}_{\mathcal{T}}$ we are considering.

Recall from linear algebra that $\dim(V) = \dim(V^*)$ holds for any finite-dimensional vector space V, where V^* denotes the dual of V (consisting of all linear forms $f : V \to \mathbb{Q}$).

Theorem 4.2. *Suppose that $\mathcal{T}$ consists of squares, then the space $\mathscr{L}_{\mathcal{T}}$ is one-dimensional.*

Proof. Let $V := \mathscr{L}_{\mathcal{T}}^*$ denote the dual space and consider the linear map $\chi : V \to \mathbb{Q}$ that sends a linear form $\varphi : \mathscr{L}_{\mathcal{T}} \to \mathbb{Q}$ to the rational number $\varphi(w(R))$. By Lemma 4.1 we have $\varphi = 0$ if and only if $\varphi(w(R)) = 0$, so we find $\ker(\chi) = \{0\}$. In other words, χ is injective. Therefore we have $\dim(V) \leq 1$, hence

$$\dim\left(\mathscr{L}_{\mathcal{T}}\right) = \dim\left(\mathscr{L}_{\mathcal{T}}^*\right) \leq 1.$$

Of course we have non-zero elements in $\mathscr{L}_{\mathcal{T}}$, such as $\ell(S_1), \ldots, \ell(S_n)$ as well as $w(R)$ and $h(R)$, so we also have $\dim(\mathscr{L}_{\mathcal{T}}) \geq 1$. This proves the result. ■

Now we are basically done. The results obtained so far are summarised in the following corollary.

Corollary 4.3. *Let R be a rectangle and let $\mathcal{T} = \{S_1, \ldots, S_n\}$ be a square tiling of R. Then $w(R)$ and $h(R)$ are rational multiples of one another. Furthermore, the lenghts of the sides of the squares in the tiling are also rational multiples of $w(R)$.*

Proof. The space $\mathscr{L}_{\mathcal{T}}$ is one-dimensional and contains $w(R)$ as a non-zero element. Therefore every element of $\mathscr{L}_{\mathcal{T}}$ is a rational multiple of $w(R)$. This holds in particular for $\ell(S_1), \ldots, \ell(S_n)$ and $h(R)$. ■

Note that we have unveiled even more structure than originally asked for: not only are $w(R)$ and $h(R)$ necessarily rational multiples of one another; the sides of the squares in the tiling must also be rational multiples of $w(R)$. This shows that the square tiling can always be drawn on some regular grid consisting of squares of constant size.

5 Related problems

The techniques developed so far can also be used to solve a couple of related problems regarding rectangular tilings of rectangles. Here we no longer require the rectangles S_i in the tiling $\mathcal{T}$ to be squares.

5.1 Rectangles with rational proportions

One straightforward relaxation of the problem is the following question: what happens if we only require the individual tiles in the tiling to be rectangles with rational proportions ($\frac{h(S_i)}{w(S_i)} \in \mathbb{Q}$), not necessarily squares? A quick inspection of the proof shows that the conclusions of Corollary 4.3 still hold in this case. We only need a minor modification to the proof of Lemma 4.1. Let $r_1, \ldots, r_n \in \mathbb{Q}$ be such that $h(S_i) = r_i \cdot w(S_i)$ holds for all $i \in \{1, \ldots, n\}$, and let f be as in the proof of the aforementioned lemma. Then we have

$$f(w(S_i), h(S_i)) = \varphi(w(S_i)) \cdot \varphi(h(S_i)) = \varphi(w(S_i)) \cdot \varphi(r_i \cdot w(S_i)) = r_i \cdot \varphi(w(S_i))^2,$$

which is again non-negative. Thus, we have proven the following corollary:

Corollary 5.1. *Let R be a rectangle that can be tiled by rectangles of rational proportions. Then R is of rational proportions as well.*

This is not very surprising: if R can be tiled with rectangles of rational proportions, then every rectangle of this tiling can be tiled with squares, so R can also be tiled with squares.

5.2 Rearranging the pieces

Next, we consider the following question: given two rectangles R_1 and R_2, is it possible to cut up R_1 into finitely many rectangular pieces, translate (but not rotate!) the pieces independently, and finally glue them back together in order to form R_2? This notion is made precise in the following definition.

Definition 5.2. Suppose that $R_1, R_2 \subseteq \mathbb{R}^2$ are rectangles with sides parallel to the coordinate axes. We say that R_1 and R_2 are *equi-tileable*[2] if there exist tilings $\mathcal{S} = \{S_1, \ldots, S_n\}$ and $\mathcal{T} = \{T_1, \ldots, T_n\}$ of R_1 and R_2, respectively, such that $w(S_i) = w(T_i)$ and $h(S_i) = h(T_i)$ hold for all $i \in \{1, \ldots, n\}$.

In this setting we have the following theorem.

Theorem 5.3. *Two rectangles $R_1, R_2 \subseteq \mathbb{R}^2$ are equi-tileable if and only if there exists a rational number $r \in \mathbb{Q}_{>0}$ such that $\frac{w(R_2)}{w(R_1)} = \frac{h(R_1)}{h(R_2)} = r$ holds.*

Proof. On the one hand, suppose that $\frac{w(R_2)}{w(R_1)} = \frac{h(R_1)}{h(R_2)} = r$ holds for some positive rational number r, and write $r = \frac{a}{b}$ with $a, b \in \mathbb{Z}_{>0}$. Then both R_1 and R_2 can be tiled as rectangular grids where every elementary rectangle has width $\frac{w(R_1)}{b} = \frac{w(R_2)}{a}$ and height $\frac{h(R_1)}{a} = \frac{h(R_2)}{b}$.

Conversely, suppose that R_1 and R_2 are equi-tileable. Choose tilings $\mathcal{S} = \{S_1, \ldots, S_n\}$ and $\mathcal{T} = \{T_1, \ldots, T_n\}$ as in Definition 5.2. Note that we have $\mathscr{L}_{\mathcal{S}} = \mathscr{L}_{\mathcal{T}}$, as the vector space is spanned by the widths and heights of the individual tiles. (However, unlike in the case of square tilings, this space need not be one-dimensional now! For every $n \in \mathbb{N}^+$ it is easy to construct an example where $\mathscr{L}_{\mathcal{S}}$ has dimension n.)

By Lemma 3.2, for any $\mathcal{T}$-area function $f : \mathscr{L}_{\mathcal{T}} \times \mathscr{L}_{\mathcal{T}} \to V$ we have

$$f\big(w(R_2), h(R_2)\big) = \sum_{i=1}^{n} f\big(w(T_i), h(T_i)\big) = \sum_{i=1}^{n} f\big(w(S_i), h(S_i)\big) = f\big(w(R_1), h(R_1)\big).$$

Of course, equi-tileable rectangles must have the same area. Indeed, our normal understanding of area corresponds with the $\mathcal{T}$-area function $f : \mathscr{L}_{\mathcal{T}} \times \mathscr{L}_{\mathcal{T}} \to \mathbb{R}$ given by $f(x, y) = x \cdot y$, so that the above equality becomes $w(R_2) \cdot h(R_2) = w(R_1) \cdot h(R_1)$. This may be rewritten as

$$\frac{w(R_2)}{w(R_1)} = \frac{h(R_1)}{h(R_2)}. \tag{5.4}$$

Suppose, for the sake of contradiction, that the ratio in (5.4) is irrational. Then $w(R_1)$ and $w(R_2)$ are linearly independent over $\mathbb{Q}$. By Proposition A.1 there exists a linear form $\varphi : \mathscr{L}_{\mathcal{T}} \to \mathbb{Q}$ satisfying $\varphi(w(R_1)) = 1$ and $\varphi(w(R_2)) = 0$. Consider the $\mathcal{T}$-area function $f_\varphi : \mathscr{L}_{\mathcal{T}} \times \mathscr{L}_{\mathcal{T}} \to \mathbb{R}$ given by $f_\varphi(x, y) = \varphi(x) \cdot y$, then we have

$$0 = f_\varphi\big(w(R_2), h(R_2)\big) = f_\varphi\big(w(R_1), h(R_1)\big) = h(R_1) \neq 0. \tag{5.5}$$

This is a contradiction, so we conclude that $\frac{w(R_2)}{w(R_1)}$ must be rational. ■

Remark 5.6. Observe the strange definition of f_φ here: it is not symmetric in x and y. This choice is made because we do not know anything about $\varphi(h(R_1))$, but we want to be certain that we end up with a non-zero number in (5.5). Luckily the codomain of a $\mathcal{T}$-area function is allowed to be an arbitrary $\mathbb{Q}$-vector space V, since Lemma 3.2 makes no additional assumptions about the space V. In this case it was necessary to set $V = \mathbb{R}$ (after all, $\varphi(x)$ is rational for all $x \in \mathscr{L}_{\mathcal{T}}$, but then we multiply it by y, which is not typically rational), but this is not a problem. Similarly, the standard area function $f(x, y) = x \cdot y$ also requires us to choose $V = \mathbb{R}$.

[2] Terminology invented by the author. Suggested Dutch term: "gelijk betegelbaar".

6 Further reading

A similar, more succinct solution to the main problem is given in [AZ2014]. The exposition given there is considered to be too succinct to serve as material for SPC. The same proof also occurs in [Mat2010, Chapter 12].

The original solution to this problem was given by Max Dehn in [Deh1903].

Appendix A

The following proposition from linear algebra is very useful in the context of rectangular/square tilings. Indeed, it is used in the proof of Theorem 5.3, and also in the solution to the main problem presented in [AZ2014].

Proposition A.1. *Let V be a finite-dimensional vector space over some field $\mathbb{F}$ and suppose that $x, y \in V$ are linearly independent. Then there exists a linear form $\varphi : V \to \mathbb{F}$ satisfying $\varphi(x) = 1$ and $\varphi(y) = 0$.*

Proof. Since x and y are linearly independent, we may extend the two-element set $\{x, y\}$ to a basis $\mathcal{B} = \{b_1, b_2, \ldots, b_m\}$ of V with $b_1 = x$ and $b_2 = y$. Consider the dual basis $\mathcal{B}^* = \{b_1^*, b_2^*, \ldots, b_m^*\}$ of the dual space $V^* := \mathrm{Hom}(V, \mathbb{F})$, defined by

$$b_i^*(b_j) = \begin{cases} 1, & \text{if } i = j; \\ 0, & \text{if } i \neq j. \end{cases}$$

Then the choice of $\varphi := b_1^*$ suffices. ■

It should be pointed out that the linear form φ from Proposition A.1 is not generally unique. Indeed, there are many ways to extend the two-element set $\{x, y\}$ to a basis, and different bases result in different dual bases. (The notation of dual bases is a little unfortunate: we only use the dual element b_1^*, but this depends on the entire basis $\mathcal{B}$, not just b_1.)

As a concrete example, let us take $V = \mathbb{Q}^3$ with $x = (1, 0, 0)$ and $y = (0, 1, 0)$. If we extend this to a basis by adding the vector $z = (0, 0, 1)$, then we end up with the linear form $\varphi : V \to \mathbb{Q}$ given by $(v_1, v_2, v_3) \mapsto v_1$. However, if we choose to add the vector $\omega = (1, 1, 1)$ instead, then we end up with the linear form $\psi : V \to \mathbb{Q}$ given by $(v_1, v_2, v_3) \mapsto v_1 - v_3$. In this example, any linear form of the form $(v_1, v_2, v_3) \mapsto v_1 + \lambda v_3$ (for some $\lambda \in \mathbb{Q}$) meets the requirements, so there are infinitely many valid choices for φ.

For more information about linear forms and dual bases, please refer to the lecture notes of Linear Algebra II.

References

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